

OPERATORS APPROXIMABLE BY HYPERCYCLIC OPERATORS

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ABSTRACT. We show that operators on a separable infinite dimensional Banach space X of the form $I + S$, where S is an operator with dense generalised kernel, must lie in the norm closure of the hypercyclic operators on X , in fact in the closure of the mixing operators.

1. INTRODUCTION

Hypercyclic operators $T: X \rightarrow X$ on a Banach space X (those bounded linear operators for which there is a vector $v \in X$ with dense orbit $\{v, Tv, T^2v, \dots\}$) have been the subject of intense study in recent decades and we refer to [3] for a recent overview of the topic. Many problems that were posed about hypercyclicity have now been solved. In particular, it is well known that every separable infinite dimensional Banach space X supports hypercyclic operators and in fact many such operators. We use $\mathcal{B}(X)$ for the bounded linear operators on X (always over \mathbb{C}) and $\mathcal{HC}(X)$ for the subclass of hypercyclic operators. An elegant result of [4] allows one to conclude that, for $T \in \mathcal{HC}(X)$, the similarity class $\{S^{-1}TS : S \in \mathcal{B}(X) \text{ invertible}\}$ is always dense in the strong operator topology of $\mathcal{B}(X)$ (and consists of hypercyclic operators).

In the operator norm on $\mathcal{B}(X)$, there are some obvious restrictions such as $\|T\| > 1$ for $T \in \mathcal{HC}(X)$, but Herrero [6] characterised the norm closure $\overline{\mathcal{HC}(H)}$ of $\mathcal{HC}(H)$ in spectral terms for H a (separable infinite dimensional) Hilbert space. In fact the paper [6] may be seen as flowing from a considerable body of work (see [5, 1]) aimed at characterising the norm closure of the similarity class of a Hilbert space operator, but [6, Remark 2.3] notes that the necessary and sufficient conditions for a Hilbert space operator $T \in \mathcal{B}(H)$ to belong in $\overline{\mathcal{HC}(H)}$ are also necessary for the Banach space case. In the extreme case of pathological Banach spaces Y with few operators, the existence of which was recently established by Argyros and Haydon [2], all operators $T \in \mathcal{B}(Y)$ are of the form $T = \lambda I + K$ with λ a scalar and K compact, and

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then the Herero conditions reduce to $|\lambda| = 1$ and K quasinilpotent (spectrum just $\{0\}$).

Our main result (Theorem 3.1) implies (Corollary 3.2) that (for X a separable infinite dimensional Banach space) operators $T = \lambda I + N$ with $|\lambda| = 1$ and $N \in \mathcal{B}(X)$ nilpotent are all in the operator norm closure of $\mathcal{HC}(X)$. So far we have been unable to decide whether this result remains true with N replaced by a quasinilpotent K , but we have some results for special cases of quasinilpotent K .

The methods we use to obtain this approximation are largely elementary, based on certain notions of generalised shifts — which may be seen as operators that admit a Jordan form representation in a somewhat loose sense. We rely then on an observation of V. Müller [8, Theorem 1] which is a simplification of [3, Theorem 2.2], attributed to Grivaux and Shkarin, to show that translates of such shifts by the identity operator I , or by λI with $|\lambda| = 1$, are mixing (and hence hypercyclic).

We recall that $T \in \mathcal{B}(X)$ is called a *mixing operator* if for each pair U, V of nonempty open subsets of X , there is N such that $T^n(U) \cap V \neq \emptyset \forall n > N$. We will use $\mathcal{Mix}(X)$ to denote the mixing operators on X .

2. GENERALISED SHIFTS

For $T \in \mathcal{B}(X)$ we use the notation $\mathcal{N}^\infty(T)$ for the generalised kernel $\bigcup_{n=1}^\infty \ker(T^n)$ and $\mathcal{R}^\infty(T) = \bigcap_{n=1}^\infty T^n(X)$ for the hyperrange.

It will be convenient for us to introduce the (lexicographic) ordering $<$ on $\mathbb{N} \times \mathbb{N}$ given by $(p, q) < (\ell, k)$ if either $p < \ell$ or $p = \ell$ and $q < k$. Given $(\ell, k) \in \mathbb{N} \times \mathbb{N}$ with $k > 1$ we refer to $(\ell, k-1)$ as the *immediate predecessor* of (ℓ, k) . This helps to explain the notions of generalised (backward) shift that we will use. Notice that we base these notions of shifts on sets of vectors which span a dense linear subspace, but need not be Schauder bases for X .

Definition 2.1. Let X be a separable infinite dimensional Banach space. Given a set of linearly independent vectors $\{x_k^\ell : (\ell, k) \in \{1, \dots, n\} \times \mathbb{N}\}$ whose span is dense in X , we call $S \in \mathcal{B}(X)$ a *generalised backward n -shift adapted to the set* if for each $(\ell, k) \in \{1, \dots, n\} \times \mathbb{N}$, $T(x_k^\ell)$ is a finite linear combination of x_q^p with $(p, q) < (\ell, k)$ and is such that the coefficient of the immediate predecessor, x_{k-1}^ℓ , is non-zero when $k > 1$. In more formal terms, we insist that

$$(2.1) \quad Sx_k^\ell = \sum_{i=1}^{k-1} a_i^{\ell,k} x_i^\ell + \sum_{i=1}^{\ell-1} \sum_{j=1}^{m_j^{\ell,k}} b_{i,j}^{\ell,k} x_j^i \quad a_i^{\ell,k}, b_{i,j}^{\ell,k} \in \mathbb{K},$$

$$a_{k-1}^{\ell,k} \neq 0 \text{ if } k > 1, m_j^{\ell,k} \in \mathbb{N}.$$

Similarly, given a set of linearly independent vectors $\{x_k^\ell : (\ell, k) \in \mathbb{N} \times \mathbb{N}\}$ whose span is dense in X , we call $S \in \mathcal{B}(X)$ a *generalised*

backward ∞ -shift (adapted to the set) if for each $(\ell, k) \in \mathbb{N} \times \mathbb{N}$, $T(x_k^\ell)$ is a finite linear combination (2.1).

We think of the sequence $(x_k^\ell)_{k=1}^\infty$ as the ℓ^{th} chain of the set.

Lemma 2.2. *If $S \in \mathcal{B}(X)$ is a generalised n -shift or ∞ -shift adapted to a set $\{x_k^\ell\}$, then $x_k^\ell \in \mathcal{R}^\infty(S) \cap \mathcal{N}^\infty(S)$ (for each (ℓ, k) in the set).*

Proof. The proof is a simple induction on ℓ .

For $\ell = 1$, $Sx_k^1 = \sum_{i=1}^{k-1} a_i^{1,k} x_i^1 \in \text{span}\{x_i^1 : 1 \leq i \leq k-1\}$, $x_1^1 \in \ker S$. Clearly $x_k^1 \in \ker S^k \subseteq \mathcal{N}^\infty(S)$ for each k . Also,

$$S^r \left(\frac{x_{\ell+1}^1}{\prod_{i=1}^r a_i^{1,i+1}} \right) = x_1^1$$

and so $x_1^1 \in S^r(X)$ for all r . Hence $\mathbb{C}x_1^1 \subseteq \mathcal{R}^\infty(S)$. Using a subinduction on k , assume $x_i^1 \in \mathcal{R}^\infty(S)$ for $1 \leq i \leq k$. Then

$$S^r \left(\frac{x_{r+k+1}^1}{\prod_{i=1}^r a_{i+k}^{1,i+k+1}} \right) = x_{k+1}^1 + u$$

with $u \in \text{span}\{x_i^1 : 1 \leq i \leq k\} \subseteq \mathcal{R}^\infty(S)$ and $u \in S^r(X) \Rightarrow x_{k+1}^1 \in S^r(X)$ for $r \geq 1$. Hence $x_{k+1}^1 \in \mathcal{R}^\infty(S)$. By induction we get $x_k^1 \in \mathcal{R}^\infty(S)$ for all k .

Similar arguments (using finiteness of the sums (2.1)) allow us to show the induction step on ℓ . \square

The following is a simplified version of [8, Theorem 1] which is a simplification of [3, Theorem 2.2].

Theorem 2.3. *Let X be a separable Banach space, $S \in \mathcal{B}(X)$ with $\mathcal{N}^\infty(S) \cap \mathcal{R}^\infty(S)$ dense in X and let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $\lambda + S \in \text{Mix}(X)$.*

Corollary 2.4. *If X is a Banach space, $S \in \mathcal{B}(X)$ a generalised n -shift or ∞ -shift (adapted to some set $\{x_k^\ell\}$) and if $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then $\lambda + S \in \text{Mix}(X)$.*

Proof. This follows from Theorem 2.3 and Lemma 2.2 \square

3. APPROXIMATING OPERATORS WITH DENSE GENERALISED KERNEL

Our aim now is to show that if an operator $S \in \mathcal{B}(X)$ (with X as usual a separable infinite dimensional Banach space) has dense generalised kernel then it can be approximated in norm by generalised shifts (as defined in Definition 2.1). Our approach is to first find linearly independent vectors

$$\{y_j^\ell : 1 \leq \ell < \infty, 1 \leq j \leq d_\ell\}$$

with dense linear span in X which are well adapted to S in a sense somewhat similar to a forward shift version of (2.1). There will however

be some important differences between what we can achieve for general S and what is required in (2.1). In particular, the vectors y_j^ℓ will not necessarily belong to $\mathcal{R}^\infty(S)$ and our approach will be to adjust the operator S (with an adjustment of at most a prescribed $\varepsilon > 0$ in norm) so as to be able to apply Corollary 2.4.

For any sequence or set of vectors $\{x_n\}$ we will use square brackets $[x_n]$ for the closure of their linear span.

We describe our construction as an algorithm, starting from a fixed sequence $\{x_n : n \in \mathbb{N}\} \subset \mathcal{N}^\infty(S)$ so that $[x_n : n \in \mathbb{N}] = X$ (which certainly exists since X is assumed to be separable). We suppose also that $\{x_n : n \in \mathbb{N}\}$ is algebraically linearly independent (or at least that $x_1 \neq 0$).

We begin by setting

$$(3.1) \quad y_1^1 = x_1.$$

Then there is a smallest $r \geq 1$ with

$$S^r y_1^1 = 0.$$

and we choose $d_1 = r$. Then the space $E_1 = \text{span}\{x_1, Sx_1, \dots, S^{r-1}x_1\}$ is S -invariant and we let

$$(3.2) \quad y_j^1 = S^{j-1}y_1^1 \quad (2 \leq j \leq d_1).$$

Assuming that we have already chosen $d_1, \dots, d_\ell \in \mathbb{N}$ and vectors $\{y_j^i : 1 \leq i \leq \ell, 1 \leq j < d_i + 1\}$, we will use the notation

$$(3.3) \quad E_m = [y_j^i : 1 \leq i \leq m, 1 \leq j \leq d_i]$$

(even for $m = 0$ when $E_m = \{0\}$) and insist during the algorithm that $S(E_m) \subseteq E_m$ holds for $m = 1, 2, \dots, \ell$ as well as

$$(3.4) \quad Sy_j^m = y_{j+1}^m \quad (1 \leq m \leq \ell, 1 \leq j < d_m),$$

$$(3.5) \quad Sy_{d_m}^m \in E_{m-1} \quad (1 \leq m \leq \ell),$$

and

$$(3.6) \quad \text{span}\{y_j^m : 1 \leq j \leq d_m\} \cap E_{m-1} = \{0\} \quad (1 \leq m \leq \ell).$$

We will have in addition that

$$(3.7) \quad \{x_1, \dots, x_m\} \subset E_m \quad (1 \leq m \leq \ell),$$

$$(3.8) \quad y_1^m \in \{x_1, x_2, \dots\} \quad (1 \leq m \leq \ell)$$

and

$$(3.9) \quad \{y_j^m : 1 \leq i \leq \ell, 1 \leq j \leq d_m\} \text{ linearly independent.}$$

Notice that the construction so far has ensured that these conditions hold initially for $\ell = 1$.

The next step is to choose the smallest n such that $x_n \notin E_\ell$ and to put $y_1^{\ell+1} = x_n$ (which will ensure that (3.7) and (3.8) hold for $m = \ell + 1$) and then there is a smallest $r \geq 1$ such that

$$(3.10) \quad S^{r+1}y_1^{\ell+1} \in \text{span}\{y_1^{\ell+1}, Sy_1^{\ell+1}, S^2y_1^{\ell+1}, \dots, S^ry_1^{\ell+1}\} + E_\ell \quad (\forall r \in \mathbb{N}),$$

We put $d_{\ell+1} = r$. Then we take

$$y_{j+1}^{\ell+1} = S^j y_1^{\ell+1} \quad (1 \leq j < d_{\ell+1}).$$

It is clear then that (3.4) and (3.6) hold for $m = \ell + 1$. To show (3.5), we can consider the operator induced by S on the quotient X/E_ℓ , which we temporarily denote \tilde{S} . Writing \tilde{x} for the coset $x + E_\ell$ we then have $\tilde{S}(\tilde{x}) = \widetilde{Sx}$ and \tilde{S} is a nilpotent operator with an invariant subspace $\text{span}\{\widetilde{y_j^{\ell+1}} : 1 \leq j \leq r\}$ of dimension r . Hence $\tilde{S}^r = 0$ on this subspace and (3.5) holds for $m = \ell + 1$, so that also $T(E_{\ell+1}) \subseteq E_{\ell+1}$. Finally, (3.9) is easy to check for $m = \ell + 1$.

Theorem 3.1. *Let X be a separable infinite dimensional Banach space and let $S \in \mathcal{B}(X)$ be such that $\mathcal{N}^\infty(S)$ is dense in X . Then S is in the norm closure of the generalised backward 1-shift operators on X so $T = \lambda + S \in \overline{\text{Mix}(X)}$ (the norm closure of $\text{Mix}(X)$) for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

Proof. Let $\{x_n\}_{n=1}^\infty \subset \mathcal{N}^\infty(T)$ be a linearly independent set whose span is dense in X . We now apply the above construction to produce vectors $\{y_j^\ell : 1 \leq \ell < L + 1, 1 \leq j \leq d_\ell\}$ which satisfy the conditions (3.4) through (3.9) for all $\ell < \infty$.

For each (ℓ, j) with $1 \leq j \leq d_\ell$, we can use the Hahn-Banach theorem and choose $y_j^{\ell*} \in X^*$ (the dual space of X) so that

$$y_j^{\ell*}(y_k^m) = \delta_{\ell,m} \delta_{j,k} \text{ if } m \leq \ell.$$

Given any $\varepsilon > 0$, we then choose $\varepsilon_\ell > 0$ such that

$$\sum_{\ell=1}^{\infty} \varepsilon_\ell \|y_1^{\ell+1*}\| \|y_{d_\ell}^\ell\| < \varepsilon.$$

We now define $S_\ell : X \rightarrow X$ for $\ell \geq 2$ by $S_\ell = 0$ if $y_1^{\ell-1*}(Sy_{d_\ell}^\ell) \neq 0$ and $S_\ell(x) = \varepsilon_\ell y_1^{\ell-1*}(x) y_{d_\ell}^\ell$ otherwise. Then $\|S_\ell\| \leq \varepsilon_\ell \|y_1^{\ell-1*}\| \|y_{d_\ell}^\ell\|$ and so $\sum_{\ell=1}^{\infty} \|S_\ell\| < \varepsilon$. Put $S' = S + \sum_{\ell=2}^{\infty} S_\ell$.

We claim that S' is then a generalised 1-shift in the sense of Definition 2.1 with respect to the set obtained by linearly ordering $\{y_j^\ell : \ell \in \mathbb{N}, 1 \leq j \leq d_\ell\}$ by taking those vectors in the order

$$y_{d_1}^1, \dots, y_1^1, y_{d_2}^2, \dots, y_1^2, y_{d_3}^3, \dots$$

To verify this notice that for $(\ell, 1) \leq (k, j)$ we have $S_\ell(y_j^k) \in E_{\ell-1}$ but $S_\ell(y_j^k) = 0$ if $\ell > k$. Hence $S'(y_j^k) - y_{j+1}^k \in E_{k-1}$ if $j < d_k$ by (3.4) while $S'(y_j^k) \in E_{k-1}$ if $j = d_k$ by (3.5). We have organised that

$y_1^{\ell-1,*}(S'y_{d_\ell}^\ell) \neq 0$ for $\ell \geq 2$, which is the remaining condition needed for S' to be a generalised 1-shift.

Since, for $|\lambda| = 1$, we have $\lambda + S' \in \mathcal{M}\text{ix}(X)$ by Corollary 2.4, and $\|(\lambda + S) - (\lambda + S')\| < \varepsilon$, we conclude that $\lambda + S' \in \overline{\mathcal{M}\text{ix}(X)}$ as required. \square

Corollary 3.2. *Let X be a separable infinite dimensional Banach space and let $N \in \mathcal{B}(X)$ be a nilpotent. Then $\lambda + N \in \overline{\mathcal{M}\text{ix}(X)}$.*

We say that a sequence of vectors $\{x_n\}_{n=1}^\infty$ is *minimal* if there exist functionals $x_n^* \in X^*$ such that $x_n^*(x_m) = \delta_{n,m}$ and a minimal sequence of vectors is *fundamental* if its linear span is dense in X . The x_n^* are known as biorthogonal functionals for $\{x_n\}_{n=1}^\infty$. We call $\{x_n\}_{n=1}^\infty$ *normalised* if $\|x_n\| = 1$ for each n . In particular a Schauder basis is a minimal fundamental sequence.

By forward shift with respect to a basis $\{x_n\}_{n=1}^\infty$ we mean an operator $S \in \mathcal{B}(X)$ satisfying $Sx_n = w_n x_{n+1}$ for $n \geq 1$, where $w_n \in \mathbb{C}$ are weights. We will also consider bilateral shifts when the basis is $\{x_n\}_{n=-\infty}^\infty$.

In the next Proposition we consider a more general type of operator akin to a mixture of shifts but we do not insist on a Schauder basis.

Proposition 3.3. *Let X be a separable infinite dimensional Banach space. Suppose $S \in B(X)$ is an operator such that*

$$Sx_j^i = a_j^i x_{j+1}^i \text{ and } Sy_j^i = b_j^i y_{j+1}^i$$

where $a_j^i, b_j^i \in \mathbb{K}$ and

$\{x_j^i : 1 \leq i < I_1 + 1, 1 \leq j < \infty\} \cup \{y_j^i : 1 \leq i < I_2 + 1, -\infty < j < \infty\}$ is a normalised minimal fundamental set for X with biorthogonal functionals

$$\{x_j^{i*} : 1 \leq i < I_1 + 1, 1 \leq j < \infty\} \cup \{y_j^{i*} : 1 \leq i < I_2 + 1, -\infty < j < \infty\}$$

where $0 \leq I_1, I_2 \leq \infty$ (but we can't have $I_1 = I_2 = 0$).

If for each i with $1 \leq i < I_1 + 1$ there is an increasing sequence n_j^i with $|a_{n_j^i}^i| \|x_{n_j^i}^{i*}\| \rightarrow 0$ and for each i with $1 \leq i < I_2 + 1$ there is an increasing sequence m_j^i with $|b_{m_j^i}^i| \|y_{m_j^i}^{i*}\| \rightarrow 0$ then $\lambda + S$ is in the closure of the mixing operators.

Proof. Let $\epsilon > 0$ be given and let $\epsilon_j^i > 0, \delta_j^i > 0$ be such that

$$\sum_{i=1}^{I_1} \sum_{j=1}^{\infty} \epsilon_j^i + \sum_{i=1}^{I_2} \sum_{j=1}^{\infty} \delta_j^i < \epsilon/2.$$

We have that $|a_{n_j^i}^i| \|x_{n_j^i}^{i*}\| \rightarrow 0$ so for each i choose a subsequence $n_{k_j}^i$ such that $|a_{n_{k_j}^i}^i| \|x_{n_{k_j}^i}^{i*}\| < \epsilon_j^i$ and similarly choose a subsequence $m_{k_j}^i$

such that $|b_{m_{k_j}^i}^{i*}| \|y_{m_{k_j}^i}^{i*}\| < \delta_j^i$. Let

$$K = \sum_{i=1}^{I_1} \sum_{j=1}^{\infty} a_{n_{k_j}^i}^i x_{n_{k_j}^i}^{i*} \otimes x_{n_{k_j}^i+1}^i + \sum_{i=1}^{I_2} \sum_{j=1}^{\infty} b_{m_{k_j}^i}^i y_{m_{k_j}^i}^{i*} \otimes y_{m_{k_j}^i+1}^i.$$

Then

$$\begin{aligned} \|K\| &\leq \sum_{i=1}^{I_1} \sum_{j=1}^{\infty} |a_{n_{k_j}^i}^i| \|x_{n_{k_j}^i}^{i*}\| + \sum_{i=1}^{I_2} \sum_{j=1}^{\infty} |b_{m_{k_j}^i}^i| \|y_{m_{k_j}^i}^{i*}\| \\ &< \sum_{i=1}^{I_1} \sum_{j=1}^{\infty} \epsilon_j^i + \sum_{i=1}^{I_2} \sum_{j=1}^{\infty} \delta_j^i < \epsilon/2 \end{aligned}$$

Claim. $\mathcal{N}^\infty(S - K)$ is dense in X .

Proof. First, we will show that $x_j^i \in \mathcal{N}^\infty(S - K)$ for each i, j . Note that $(S - K)x_j^i = Sx_j^i = a_j^i x_{j+1}^i$ for $(i, j) \neq (i, n_{k_m}^i)$. We have $(S - K)x_{n_{k_j}^i}^i = a_{n_{k_j}^i}^i x_{n_{k_j}^i+1}^i - a_{n_{k_j}^i}^i x_{n_{k_j}^i+1}^i = 0$. Suppose $n_{k_j}^i < j \leq m_{k_j}^i$, say $j = m_{k_j}^i - \ell$, then $(S - K)^{\ell+1} x_j^i = (a_j^i a_{j+1}^i \dots a_{m_{k_j}^i-1}^i)(S - K)x_{j+\ell}^i = 0$. So $x_j^i \in \mathcal{N}^\infty(S - K)$.

Similarly $y_j^i \in \mathcal{N}^\infty(S - K)$ for $j \geq m_1^i$. But we also have that $(S - K)y_j^i = Sy_j^i$ for $j < m_1^i$. Let $j = m_1^i - \ell$ for some $\ell \in \mathbb{N}$. Then $(S - K)^{\ell+1} y_j^i = (b_j^i b_{j+1}^i \dots b_{m_1^i-1}^i)(S - K)y_{j+\ell}^i = 0$ so $y_j^i \in \mathcal{N}^\infty(S - K)$.

Then $\text{span}\{x_j^i\} \cup \{y_j^i\} \subset \mathcal{N}^\infty(S - K)$ and therefore $\mathcal{N}^\infty(S - K)$ is dense in X . \square

Therefore by Theorem 3.1 we have that there is a 1-shift B such that $\|S - K - B\| < \epsilon/2$. So $\|S - B\| \leq \|S - (S - K)\| + \|(S - K) - B\| < \epsilon$ and therefore (via Corollary 2.4) $\lambda + S$ is in the closure of the mixing operators. \square

Corollary 3.4. *Let X be a separable infinite dimensional Banach space with a Schauder basis. If S is a weighted forward or bilateral shift on a normalised basis for X with weights w_j such that there exists an increasing sequence n_j with $|w_{n_j}| \rightarrow 0$ then $\lambda + S$ is in the closure of the mixing operators.*

In particular if S is a quasinilpotent forward or bilateral shift on a normalised basis then $\lambda + S$ is in the closure of the mixing operators.

Proof. Suppose e_n is our normalised basis. By a remark at the beginning of [7, 1.b] we have that $\sup_n \|e_n^*\| = K < \infty$. Then $|w_{n_j}| \|e_n^*\| \leq K|w_{n_j}| \rightarrow 0$ and the rest follows from Proposition 3.3 by setting $I_1 = 1$,

$I_2 = 0$ and $x_j^1 = e_j$ for the forward shift case and $I_1 = 0$, $I_2 = 1$ and $y_j^1 = e_j$ for the bilateral shift case.

For the second part we need only show that if the shift is quasinilpotent then there must be a subsequence of the weights going to 0. Assume for the sake of contradiction that this is not true. Then there is $j \in \mathbb{N}$ such that $C = \inf_{n \geq j} \{|w_n|\} \neq 0$. Choose $0 < \epsilon < C$. Then there is $N \in \mathbb{N}$ such that $\|S^n x\| < \epsilon^n \|x\|$ for all $n > N$.

$$C^n \leq \|w_j w_{j+1} \dots w_{j+n-1} e_{j+n}\| = \|S^n e_j\| < \epsilon^n < C^n$$

for $n > N$.

□

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